Formalising Pédrot's Prefascist CwF with Π-types in Agda

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Abstract

We study Pédrot's prefascist sets as an alternative presentation of presheaves in intensional type theory. Following Kaposi and Pujet's work, we formalise the prefascist CwF_Π in Agda using a metatheory inspired by observational type theory. All the resulting CwF_Π equations hold definitionally except for the η-rule of context comprehension. Possible extensions of the formalisation are also discussed.

Introduction Given a base category $\mathcal{C}$, a *presheaf* over $\mathcal{C}$ is simply a functor $\Gamma : \mathcal{C}^{\mathrm{op}} \to \mathrm{Set}$. Presheaves over $\mathcal{C}$ and natural transformations between them form a category $\mathrm{Psh}(\mathcal{C})$. Presheaf models of type theory appear ubiquitously in the literature, for example in [Hof97, Hub19, Péd20]. From now on, we assume that all categorical equations of $\mathcal{C}$ hold definitionally[1].

There had long been interest in formalising presheaves solely within type theory, but it was not until Pédrot's work [Péd20] that a purely syntactic presheaf model of CIC was given which preserves conversion while supporting full dependent elimination. This construction induces objects originally called "prefascist sets", which can be thought of as better representations of presheaves in intensional type theory, since their functoriality and naturality equations hold definitionally [KP25a, §5.2]. Together with morphisms between them, prefascist sets form a category $\mathrm{Pfs}(\mathcal{C})$, which is equivalent to $\mathrm{Psh}(\mathcal{C})$, assuming function extensionality [Péd20, §7] [KP25a, §5.1]. Moreover, $\mathrm{Pfs}(\mathcal{C})$ has a *strict CwF structure with fully strict* Π*-types*: all its CwF_Π equations are definitional except for the η-rule of context comprehension [KP25a, §5.2].

Kaposi and Pujet's Formalisation As part of their broader strictification project, Kaposi and Pujet [KP25a] gave an Agda formalisation of the prefascist CwF_Π using a two-level type theory as its metatheory. The inner layer is a form of observational type theory (OTT), corresponding to the theory implemented by Agda, while the outer layer is extensional type theory, which is used to reason externally about the inner layer. The inner universe is denoted by Set[2], which contains a subuniverse of strict propositions $\mathsf{Prop} : \mathsf{Set}$, i.e. any two terms in a strict proposition are definitionally equal. Moreover, every type $A : \mathsf{Set}$ is equipped with a Prop-valued observational equality relation, with constructor refl and transport [KP25a]. To implement this in Agda[3], they define Prop-valued homogeneous $(_ \equiv _)$ and heterogeneous equalities, using Agda's Prop hierarchy. They postulate a type coercion operator $\mathsf{coe}_{\equiv} : A \equiv B \to A \to B$ and its computation rule $\mathsf{coe}_{\equiv}\ e\ u \equiv u$ for $e : A \equiv A$, which is registered as a rewrite rule [KP25a, §1.5] [KP25b]. Their library additionally postulates heterogeneous function extensionality and some other principles [KP25b], which support their complete strictification development [KP25a].

[1] If this is not the case, we may replace $\mathcal{C}$ by its image under the Yoneda embedding, whose categorical equations are definitional. For details, see [KP25a, §5.3].

[2] Kaposi and Pujet use U for the inner universe and Set for the outer universe. We deliberately change the notation to match the Agda formalisation.

[3] Note that Agda does not support OTT natively.

The definitions and motivations of the four sorts Con, Sub, Ty, and Tm, together with context comprehension and Π-types, are well explained in [KP25a, §§ 5.1–5.2]. The detailed constructions of the projections p and q, pairing, λ-abstraction, and application are given in [KP25b]. All the resulting CwF_Π equations hold definitionally except for the η-rule of context comprehension [KP25a, §5.2] [KP25b]. We refer interested readers to these sources for details.

Main Contribution Our project aims at understanding the prefascist model by formalisation. Following [KP25a], we formalise the prefascist CwF_Π in Agda using a smaller OTT-inspired metatheory. We highlight the important parts of our formalisation.

Our metatheory is ordinary Agda together with three postulates: funext, coe, and $\mathsf{coe\text{-}refl}$. The type of funext is standard, while our coercion is stated directly as dependent transport, $\mathsf{coe} : (P : A \to \mathsf{Set}) \to x \equiv y \to P\,x \to P\,y$, where the equality is Prop-valued. This operation and $\mathsf{coe}_{\equiv}$ are interdefinable[4]. The postulate $\mathsf{coe\text{-}refl}$ is the computation rule $\mathsf{coe}\,P\,\mathsf{refl}\,u \equiv u$, which is registered as a rewrite rule. These postulates resemble the fragment of OTT needed by our construction, but our metatheory has no heterogeneous equality. Thus, the funext here is homogeneous function extensionality. We also define $\mathsf{coe}^{\mathsf{p}}$ and derive $\mathsf{funext}_{\mathsf{i}}$ from funext; these are, respectively, the propositional counterpart of coe and the counterpart of funext for functions with implicit arguments. Finally, all categorical equations of the base category are registered as local rewrite rules, so Agda treats them definitionally within the construction.

The construction itself follows [KP25a] closely. Its main technical points are context comprehension and Π-types. We begin with context comprehension. At every restriction, a raw thunk θ in $\Gamma \triangleright A$ contains a context component and a dependent component. A coherence proof e shows that the first components assemble into an element of $\mathsf{El}\,\Gamma\,x$. The second components are then transported to form a dependent element over it, which is required to be valid in A. Below, Σ is Agda's built-in dependent pair, while we define $\exists$ as its Prop-counterpart:

$$\begin{aligned}
_ \triangleright _ &: (\Gamma : \mathsf{Con}) \to (A : \mathsf{Ty}\,\Gamma) \to \mathsf{Con},\\
(\Gamma \triangleright A)_0\,x &\coloneqq \Sigma\,(\gamma : \mathsf{El}\,\Gamma\,x)\,(A_0\,\gamma),\\
(\Gamma \triangleright A)_1\,\theta &\coloneqq \exists\,(e : \Gamma\text{-}\mathrm{coh}\,\theta)\,(A\text{-}\mathrm{valid}\,\theta\,e).
\end{aligned}$$

For Π-types, the main difficulty is that $\mathsf{ElT}\,A\,\gamma$ is defined as an Agda record. If Pi quantifies directly over $a : \mathsf{ElT}\,A\,\gamma$, Agda does not identify the two resulting input types definitionally after substitution, so the two sides of the substitution law are not definitionally equal. We solve this by expanding ElT into its two fields. The type $\mathsf{RunT}\,A\,\gamma$ describes the computational component, while $\mathsf{SatT}\,A\,\gamma\,r$ states its validity; $\mathsf{packT}\,r\,s$ packages them back into an element of $\mathsf{ElT}\,A\,\gamma$. We then define $(\mathsf{Pi}\,A\,B)_0\,\gamma \coloneqq (r : \mathsf{RunT}\,A\,\gamma) \to (s : \mathsf{SatT}\,A\,\gamma\,r) \to B_0(\gamma, \mathsf{packT}\,r\,s)$. This is an expanded version of the construction in [KP25a, §5.2]: Kaposi and Pujet quantify directly over the corresponding record and state Pi-substitution using heterogeneous equality [KP25b], whereas our construction makes the same law definitional with homogeneous equality.

Conclusion and Further Work We successfully formalise the strict prefascist CwF with fully strict Π-types, showing that our smaller OTT-inspired metatheory suffices. Further work includes more type formers such as Booleans, whose recipe is in [KP25a], and identity types. When the base category $\mathcal{C}$ is itself a CwF, the Yoneda embedding $ɏ_{\mathcal{C}} : \mathcal{C} \to \mathrm{Psh}(\mathcal{C})$ extends to a weak morphism of CwFs from $\mathcal{C}$ into the induced inner CwF on $\mathrm{Psh}(\mathcal{C})$ [ACKS23, §2.5.3]. We have also formalised the prefascist Yoneda embedding $\exists_{\mathcal{C}} : \mathcal{C} \to \mathrm{Pfs}(\mathcal{C})$; extending it to the corresponding weak CwF morphism in the prefascist presentation is another future direction.

[4] coe specialises to $\mathsf{coe}_{\equiv}$ by taking $P = \lambda X.\,X$, while $\mathsf{coe}_{\equiv}$ gives dependent transport along $\mathsf{cong}\,P\,e$.

References

[ACKS23] Danil Annenkov, Paolo Capriotti, Nicolai Kraus, and Christian Sattler. Two-level type theory and applications. *Mathematical Structures in Computer Science*, 33(8):688–743, 2023.

[Hof97] Martin Hofmann. Syntax and semantics of dependent types. In Andrew M. Pitts and Peter Dybjer, editors, *Semantics and Logics of Computation*, pages 79–130. Cambridge University Press, 1997.

[Hub19] Simon Huber. Canonicity for cubical type theory. *Journal of Automated Reasoning*, 63(2):173–210, 2019.

[KP25a] Ambrus Kaposi and Loïc Pujet. Type theory in type theory using a strictified syntax. *Proceedings of the ACM on Programming Languages*, 9(ICFP), 2025.

[KP25b] Ambrus Kaposi and Loïc Pujet. Type theory in type theory using a strictified syntax: Accompanying formalisation, July 2025. Agda formalisation.

[Péd20] Pierre-Marie Pédrot. Russian constructivism in a prefascist theory. In *Proceedings of the 35th Annual ACM/IEEE Symposium on Logic in Computer Science*, 2020.